

DECOMPOSING GORENSTEIN RINGS AS CONNECTED SUMS

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ABSTRACT. In [2], Ananthnarayan, Avramov and Moore give a new construction of Gorenstein rings from two Gorenstein local rings, called their connected sum. Given a Gorenstein ring, one would like to know whether it can be decomposed as a connected sum and if so, what are its components. We answer these questions in the case of an Gorenstein Artin local algebra over a field, and give a characterization for such rings to be connected sums. We further investigate conditions on the associated graded ring of a Gorenstein Artin ring which force it to be a connected sum, and as a consequence, obtain results about its Poincaré series and minimal number of generators of its defining ideal.

INTRODUCTION

The main object of study in this paper is a construction of Gorenstein rings, called a connected sum, defined by Ananthnarayan, Avramov and Moore in [2]. Given Cohen-Macaulay local rings R , S and \mathbf{k} of the same dimension, and ring homomorphisms $R \xrightarrow{\varepsilon_R} \mathbf{k} \xleftarrow{\varepsilon_S} S$, the authors consider the fibre product (or pullback) $R \times_{\mathbf{k}} S = \{(r, s) \in R \times S : \varepsilon_R(r) = \varepsilon_S(s)\}$ and define a connected sum of R and S over \mathbf{k} as an appropriate quotient of $R \times_{\mathbf{k}} S$. They prove that when R and S are Gorenstein, a connected sum is also a Gorenstein local ring of the same dimension.

In this paper, we focus on connected sums over a field in the Artinian case, i.e., when R and S are Gorenstein Artinian local rings and \mathbf{k} is their common residue field. These objects have been studied from different perspectives by various authors starting with Sah (cf. [13]) in the graded case, and in the local case, by Lescot (see Theorem 12). A topologically influenced version was also studied by Smith and Stong (cf. [15, Section 4]), and quite a few authors approach this area via *Macaulay's inverse systems*, (e.g., see [4]). Completely different techniques are used in this article: we look at intrinsic properties of the ring and its defining ideal in Section 3, and in Section 4, focus on conditions on the associated graded ring which force it to be a connected sum.

An important question is: Given a Gorenstein Artin ring Q , can it be decomposed as a connected sum? This question was studied from a geometric point of view for *projective bundle ideals* by Smith and Stong in [15, Section 4]. It is also known (see [2, Theorem 8.3]) that if Q has a non-trivial decomposition as a connected sum over \mathbf{k} and if the embedding dimension of Q is greater than 2, then Q cannot be a *complete intersection*. The question of decomposability has also been studied using the inverse systems by the authors in [4] using polynomials which are *direct sums* and corresponding *apolar* Gorenstein algebras (see Remark 16).

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In Section 3, we study properties of fibre products and connected sums of algebras over a field, and give two other necessary conditions for Q to be decomposable as a non-trivial connected sum: one concerning the second Hilbert coefficient $H_Q(2)$ of Q (Theorem 20), and the second in terms of the associated graded ring $\text{gr}(Q)$ of Q (Theorem 26). In particular, if Q is a *compressed Gorenstein algebra* over k with Loewy length at least 4, then Q is indecomposable as a connected sum.

Finally, we identify conditions under which a Gorenstein Artin local algebra over a field is a connected sum in Proposition 27 and obtain a characterization for the same in terms of the defining ideals (see Theorem 31). A secondary question is: If Q is a connected sum, what are its components? In general, it is not clear how to extract the components of a connected sum, but this characterization allows us to do so in this case (see Proposition 29).

In Section 4, we study the associated graded ring of a Gorenstein Artin ring and look for conditions which force the given Gorenstein ring to be a connected sum. The motivation comes from the following: In [14], Sally proves a structure theorem for stretched Gorenstein rings and in [7], Elias and Rossi give a similar structure theorem for short Gorenstein rings with some assumptions on the residue field. When such a short or stretched Gorenstein ring Q is an algebra over a field, these structure theorems show that Q can be written as a connected sum of a graded Gorenstein Artin ring R with the same Loewy length as Q , and a Gorenstein Artin ring S with Loewy length less than three.

In either case, using a construction of Iarrobino, we see that the associated graded ring of Q has the property that its socle in degrees two or higher is one-dimensional. We call such graded k -algebras *Gorenstein up to linear socle*. Using the characterization proved in Section 3, we show that this property allows us to decompose Q as a connected sum, where one of its components has Loewy length less than three and the other has a Gorenstein associated graded ring. This gives us results regarding the Poincaré series of Q and minimal number of generators of its defining ideal. (See Theorem 39 and its corollaries).

In Section 5, we use Theorem 39 to give applications to short and stretched Gorenstein k -algebras. In particular, we show that these rings, when they are not graded, are non-trivial connected sums, and derive some consequences without any restrictions on the residue field.

The first two sections contain results regarding the main tools that are used in the rest of the paper. In Section 1, we list some properties of associated graded rings and Poincaré series. Section 2 contains some information on fibre products and connected sums, including their interactions with the objects introduced in Section 1. The results in Section 1 are well-known and Section 2 includes known results rephrased in our notation.

1. PRELIMINARIES

In this section, we first introduce some notation and terminology, and recall the definitions and some basic properties of the main objects that appear in this article. We also list the previously known results and some consequences that are needed in this paper.

1.1. Notation.

a) If T is a local ring and M is an T -module, $\lambda(M)$ and $\mu(M)$ respectively denote the *length* and the *minimal number of generators* of M as a T -module.

b) Let $(T, \mathfrak{m}, \mathbf{k})$ be an Artinian local ring. Then $\text{edim}(T)$ denotes the *embedding dimension* of T which is equal to $\mu(\mathfrak{m})$, and the *socle* of T is $\text{soc}(T) = \text{ann}_T(\mathfrak{m})$. Moreover, the *type* of T is $\text{type}(T) = \dim_{\mathbf{k}}(\text{soc}(T))$, and the *Loewy length* of T is $\text{ll}(T) = \max\{n : \mathfrak{m}^n \neq 0\}$.¹

c) If \mathbf{k} is a field, a *graded \mathbf{k} -algebra* G is a graded ring $G = \bigoplus_{i \geq 0} G_i$, where $G_0 = \mathbf{k}$, with unique homogeneous maximal ideal $G_+ = \bigoplus_{i \geq 1} G_i$. Furthermore, we say G is *standard graded* if G_+ is generated by G_1 .

d) Let m and n be positive integers. Then \underline{Y} and \underline{Z} denote the sets of indeterminates $\{Y_1, \dots, Y_m\}$ and $\{Z_1, \dots, Z_n\}$ respectively, and $\underline{Y} \cdot \underline{Z}$ denotes $\{Y_i Z_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.

e) Let $\iota_Y : \mathbf{k}[\underline{Y}] \hookrightarrow \mathbf{k}[\underline{Y}, \underline{Z}]$ and $\iota_Z : \mathbf{k}[\underline{Z}] \hookrightarrow \mathbf{k}[\underline{Y}, \underline{Z}]$ be the natural inclusions. For an ideal $I \subseteq \mathbf{k}[\underline{Y}, \underline{Z}]$, $I \cap \mathbf{k}[\underline{Y}]$ and $I \cap \mathbf{k}[\underline{Z}]$ denote the respective contractions $\iota_Y^{-1}(I)$ and $\iota_Z^{-1}(I)$.

Moreover, for an ideal J of $\mathbf{k}[\underline{Y}]$ or $\mathbf{k}[\underline{Z}]$, J^e denotes its extension to $\mathbf{k}[\underline{Y}, \underline{Z}]$ via the natural inclusions.

1.2. Associated Graded Rings.

Definition 1. Let $(T, \mathfrak{m}, \mathbf{k})$ be a Noetherian local ring.

i) The *graded ring associated to the maximal ideal \mathfrak{m} of P* , denoted $\text{gr}_{\mathfrak{m}}(T)$ (or simply $\text{gr}(T)$), is defined as $\text{gr}(T) \simeq \bigoplus_{i=0}^{\infty} \mathfrak{m}^i / \mathfrak{m}^{i+1}$.

ii) If $T = \bigoplus_{i \geq 0} T_i$ is a finitely generated graded \mathbf{k} -algebra, where $T_0 = \mathbf{k}$ and T_i consist of the elements in T of degree i , then we define the *Hilbert function* of T as $H_T(i) = \dim_{\mathbf{k}}(T_i)$ for $i \geq 0$. If T is not graded, we define $H_T(i) = H_{\text{gr}(T)}(i)$.

In the following remark, we list some notation and facts about associated graded rings.

Remark 2. Let $(T, \mathfrak{m}, \mathbf{k})$ be a Noetherian local ring.

a) For any $n \geq 0$, a minimal generating set of $\text{gr}(T)_n = \bigoplus_{i=n}^{\infty} \mathfrak{m}^i / \mathfrak{m}^{i+1}$, the n th power of the homogeneous maximal ideal $\text{gr}(T)_+$ of $\text{gr}(T)$, lifts to a minimal generating set of \mathfrak{m}^n .

In particular, if T is Artinian local, then so is $\text{gr}(T)$, and we have $\lambda(T) = \lambda(\text{gr}(T))$.

b) Let $x \in T$ be such that $x \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$. We define the *initial form* of x to be the element $x^* \in \text{gr}(T)$ of degree i that is the image of x in $\mathfrak{m}^i / \mathfrak{m}^{i+1}$.

c) For an ideal K of T , K^* is the ideal of $\text{gr}(T)$ defined by $\langle x^* : x \in K \rangle$. Note that, if $R \simeq T/K$, then $\text{gr}(R) \simeq \text{gr}(T)/K^*$.

We now recall a construction due to Iarrobino (cf. [9]):

Remark 3 (Iarrobino's Construction). Let $(Q, \mathfrak{m}_Q, \mathbf{k})$ be a Gorenstein Artin local ring where $\text{ll}(Q) = s$ and $G = \text{gr}(Q)$ be its associated graded ring. Iarrobino showed that

$$C = \bigoplus_{i \geq 0} \frac{(0 :_Q \mathfrak{m}_Q^{s-i}) \cap \mathfrak{m}_Q^i}{(0 :_Q \mathfrak{m}_Q^{s-i}) \cap \mathfrak{m}_Q^{i+1}}$$

is an ideal in G . He also proved that $Q_0 = G/C$ is a graded Gorenstein quotient of G with $\deg(\text{soc}(Q_0)) = s$.

Note that $H_{Q_0}(i) = H_G(i)$ for $i \geq s - 1$ since $C_i = 0$ for $i \geq s - 1$.

¹Loewy length is also referred to as *socle degree* in the literature.

1.3. Poincaré Series.

Definition 4. For a local ring $(T, \mathfrak{m}, \mathbf{k})$, the Poincaré series of T , denoted $\mathbb{P}^T(t)$, is the formal power series

$$\mathbb{P}^T(t) = \sum_{i \geq 0} \beta_i^T t^i, \quad \text{with } \beta_i^T = \dim_{\mathbf{k}} (\text{Tor}_i^T(\mathbf{k}, \mathbf{k})).$$

The Poincaré series of T can also be written as

$$\mathbb{P}^T(t) = \prod_{i \geq 1} \frac{(1 + t^{2i-1})^{\varepsilon_{2i-1}^T}}{(1 - t^{2i})^{\varepsilon_{2i}^T}},$$

(e.g., cf. [3, Remark 7.1.1]), where ε_i^T is called the i th *deviation* of T .

If T is an Artinian ring, then it is isomorphic to a quotient of a regular local ring by Cohen's Structure Theorem. If T is also a \mathbf{k} -algebra, then we can write T as a quotient of a polynomial ring over \mathbf{k} . In the following remark, we compute the minimal number of generators of its defining ideal.

Remark 5 (Minimal Number of Generators). *Let $(T, \mathfrak{m}, \mathbf{k})$ be an Artinian local ring with $\text{edim}(T) = d$.*

a) By Cohen's Structure Theorem, there is a regular local ring $(\tilde{T}, \mathfrak{m}_{\tilde{T}}, \mathbf{k})$ such that $T \simeq \tilde{T}/I_T$, where $I_T \subseteq \mathfrak{m}_{\tilde{T}}^2$. By [3, Corollary 7.1.5], $I_T \subseteq \mathfrak{m}_{\tilde{T}}^2$ implies that $\varepsilon_1^T = \text{edim}(T)$ and $\varepsilon_2^T = \mu(I_T)$.

In particular, by comparing the coefficients of t and t^2 , we get that

$$\varepsilon_1^T = \beta_1^T = d \quad \text{and} \quad \mu(I_T) = \varepsilon_2^T = \beta_2^T - \binom{\beta_1^T}{2} = \beta_2^T - \binom{d}{2}.$$

b) Suppose T is also a \mathbf{k} -algebra. We can write $T \simeq \mathbf{k}[X_1, \dots, X_d]/I_T$, where $I_T \subseteq \langle \underline{X} \rangle^2$. Since T is Artinian, $T \simeq \tilde{T}/I_T \tilde{T}$, where $\tilde{T} = \mathbf{k}[[\underline{X}]]$. Hence, by (a), $\mu(I_T) = \varepsilon_2^T = \beta_2^T - \binom{d}{2}$ in this case too.

Next we list some properties of the Poincaré series of a Gorenstein Artin local ring.

Remark 6 (Poincaré Series of Gorenstein Rings).

Let $(T, \mathfrak{m}, \mathbf{k})$ be a Gorenstein Artin local ring and \overline{T} represent the quotient $T/\text{soc}(T)$.

a) If $\text{edim}(T) \leq 3$, then $\mathbb{P}^T(t)$ is rational. The same holds for $\text{edim}(T) = 4$ if we assume $\text{char}(\mathbf{k}) \neq 2$.

Indeed, if $\text{edim}(T) = 1$, then T is a quotient of a discrete valuation ring. In particular, every ideal in T is principal. This forces $\frac{1}{\mathbb{P}^T(t)} = 1 - t$. When $\text{edim}(T) = 2$, it is a well-known result of Serre that T is a complete intersection. In particular, $\mathbb{P}^T(t)$ is rational. If $\text{edim}(T) = 3$, then $\mathbb{P}^T(t)$ is rational, for example, by [16]. Finally, by [10], if $\text{char}(\mathbf{k}) \neq 2$, then $\mathbb{P}^T(t)$ is rational if $\text{edim}(T) = 4$.

b) If $\text{edim}(T) \geq 2$, then, by [12, Theorem 2], we see that $\frac{1}{\mathbb{P}^T(t)} = \frac{1}{\mathbb{P}^{\overline{T}}(t)} + t^2$.

c) If $\text{ll}(T) = 2$ and $\text{edim}(T) = n$, it is clear that $\frac{1}{\mathbb{P}^{\overline{T}}(t)} = 1 - nt$ since $\mathfrak{m}_T \overline{T} = \text{soc}(\overline{T}) \simeq \mathbf{k}^{\oplus n}$. Therefore, by (b), $\frac{1}{\mathbb{P}^T(t)} = 1 - nt + t^2$ if $n \geq 2$. Moreover, by (a), if $\text{edim}(T) = 1$, then $\frac{1}{\mathbb{P}^T(t)} = 1 - t$. In particular, $\mathbb{P}^T(t)$ is a rational function of t .

2. FIBRE PRODUCTS AND CONNECTED SUMS

We begin with the definitions and some basic properties of fibre products and connected sums. For more details, see [2, Sections 1 and 2] and [1, Chapter 4].

Definition 7. Let $(R, \mathfrak{m}_R, \mathbf{k})$ and $(S, \mathfrak{m}_S, \mathbf{k})$ be local rings with the same residue field \mathbf{k} . We define the fibre product R and S over \mathbf{k} to be $R \times_{\mathbf{k}} S = \{(r, s) \in R \times S : \pi_R(r) = \pi_S(s)\}$, where π_R and π_S are the natural projections from R and S respectively onto \mathbf{k} .

This construction is also referred to as the pullback of the diagram $R \xrightarrow{\pi_R} \mathbf{k} \xleftarrow{\pi_S} S$.

Next we give some observations regarding fibre products.

Remark 8. With the notation as in Definition 7, set $P = R \times_{\mathbf{k}} S$. Then, we see that:

- a) $I = \{(r, 0) : r \in \mathfrak{m}_R\}$ and $J = \{(0, s) : s \in \mathfrak{m}_S\}$ are ideals of P . Identifying \mathfrak{m}_R with I and \mathfrak{m}_S with J , we can write $R \simeq P/\mathfrak{m}_S$, $S \simeq P/\mathfrak{m}_R$, $\mathbf{k} \simeq P/(\mathfrak{m}_R + \mathfrak{m}_S)$ and $\mathfrak{m}_R \cap \mathfrak{m}_S = 0$ in P .
- b) P is a local ring with unique maximal ideal $\mathfrak{m}_P = \mathfrak{m}_R \times \mathfrak{m}_S$. Furthermore, if R and S are both different from \mathbf{k} , then $\text{soc}(P) = \text{soc}(R) \oplus \text{soc}(S)$. In particular, $\text{edim}(P) = \text{edim}(R) + \text{edim}(S)$ and $\text{type}(P) = \text{type}(R) + \text{type}(S)$.
- c) If R and S are both Artinian local, then $\mathfrak{m}_P = \mathfrak{m}_R \times \mathfrak{m}_S$ shows that $\lambda(P) = \lambda(R) + \lambda(S) - 1$.

As a consequence of Remark 8(b), when R and S are Artinian local rings with $R \neq \mathbf{k} \neq S$, $P = R \times_{\mathbf{k}} S$ is Artinian, but not Gorenstein. In [2], the authors construct a suitable quotient of P which is a Gorenstein ring. They define:

Definition 9. Let $(R, \mathfrak{m}_R, \mathbf{k})$ and $(S, \mathfrak{m}_S, \mathbf{k})$ be Gorenstein Artin local rings different from \mathbf{k} . Let $\text{soc}(R) = \langle \delta_R \rangle$, $\text{soc}(S) = \langle \delta_S \rangle$. Identifying δ_R with $(\delta_R, 0)$ and δ_S with $(0, \delta_S)$, a connected sum of R and S over \mathbf{k} , denoted $R \#_{\mathbf{k}} S$, is the ring $R \#_{\mathbf{k}} S = (R \times_{\mathbf{k}} S) / \langle \delta_R - \delta_S \rangle$.

Connected sums of R and S over \mathbf{k} depend on the generators of the socle δ_R and δ_S chosen as can be seen from the following example.

Example 10. Let $R = \mathbb{Q}[Y]/\langle Y^3 \rangle$ and $S = \mathbb{Q}[Z]/\langle Z^3 \rangle$. Let y and z denote the respective images of Y and Z in R and S . Then $\text{soc}(R) = \langle y^2 \rangle$ and $\text{soc}(S) = \langle z^2 \rangle$. The connected sums $Q_1 = (R \times_{\mathbf{k}} S) / \langle y^2 - z^2 \rangle$ and $Q_2 = (R \times_{\mathbf{k}} S) / \langle y^2 - pz^2 \rangle$ are not isomorphic as rings where p is a prime number not congruent to 3 modulo 4. For a proof of this fact, see [2].

Remark 11. With notation as in Definition 9, set $P = R \times_{\mathbf{k}} S$ and let $Q = P / \langle \delta_R - \delta_S \rangle$ be a connected sum of R and S over \mathbf{k} .

- a) Note that $\lambda(Q) = \lambda(P) - 1 = \lambda(R) + \lambda(S) - 2$ since $0 \neq \delta_R - \delta_S \in \text{soc}(P)$.
- b) If $\text{ll}(R) \geq 2$ and $\text{ll}(S) \geq 2$, i.e., $\text{soc}(R) \subseteq \mathfrak{m}_R^2$ and $\text{soc}(S) \subseteq \mathfrak{m}_S^2$, then $\text{edim}(Q) = \text{edim}(P) = \text{edim}(R) + \text{edim}(S)$.
- c) By the definition of a connected sum, it is clear that $Q / \text{soc}(Q) \simeq R / \text{soc}(R) \times_{\mathbf{k}} S / \text{soc}(S)$.

A special case of Theorem 2.8 in [2] is the following, see also [11, Proposition 4.4]:

Theorem 12. Let $(R, \mathfrak{m}_R, \mathbf{k})$ and $(S, \mathfrak{m}_S, \mathbf{k})$ be Gorenstein Artin local rings different from \mathbf{k} . Then a connected sum of R and S over \mathbf{k} is also a Gorenstein Artin local ring.

The following remark tells us what the Poincaré series of fibre products and connected sums over \mathbf{k} are, in terms of the Poincaré series of the components.

Remark 13 (Poincaré Series of Fibre Products and Connected Sums).

Let $(R, \mathfrak{m}_R, \mathbf{k})$ and $(S, \mathfrak{m}_S, \mathbf{k})$ be Artinian local rings.

a) If $P = R \times_{\mathbf{k}} S$, then, by [6, Theorem 1],

$$(13.1) \quad \frac{1}{\mathbb{P}^P(t)} = \frac{1}{\mathbb{P}^R(t)} + \frac{1}{\mathbb{P}^S(t)} - 1.$$

b) Furthermore, if R and S are Gorenstein Artin, where $\text{ll}(R), \text{ll}(S) \geq 2$, then a connected sum $Q = R \#_{\mathbf{k}} S$ is Gorenstein, $\text{edim}(Q) = \text{edim}(R) + \text{edim}(S) \geq 2$ and $\overline{Q} \simeq \overline{R} \times_{\mathbf{k}} \overline{S}$. Hence $\frac{1}{\mathbb{P}^Q(t)} = \frac{1}{\mathbb{P}^R(t)} + \frac{1}{\mathbb{P}^S(t)} - 1 + t^2$. In particular, we get

$$(13.2) \quad \frac{1}{\mathbb{P}^Q(t)} = \frac{1}{\mathbb{P}^R(t)} + \frac{1}{\mathbb{P}^S(t)} - 1 + \phi(t) = \frac{1}{\mathbb{P}^P(t)} + \phi(t),$$

where $\phi(t) = -t^2$ when $\text{edim}(R) \geq 2$ and $\text{edim}(S) \geq 2$, $\phi(t) = t^2$ when $\text{edim}(R) = 1 = \text{edim}(S)$, and $\phi(t) = 0$ otherwise.

3. CONNECTED SUMS OF \mathbf{k} -ALGEBRAS

When R and S are \mathbf{k} -algebras, we can say more about their fibre products and connected sums over \mathbf{k} . We begin with the following observations:

Remark 14. Let $R = \mathbf{k}[Y_1, \dots, Y_m]/I_R$ and $S = \mathbf{k}[Z_1, \dots, Z_n]/I_S$ be \mathbf{k} -algebras (not equal to \mathbf{k}) with $I_R \subseteq \langle \underline{Y} \rangle^2$ and $I_S \subseteq \langle \underline{Z} \rangle^2$. Note that this implies $\text{edim}(R) = m$ and $\text{edim}(S) = n$.

a) Let P denote the fibre product $R \times_{\mathbf{k}} S$. Then $P \simeq \mathbf{k}[\underline{Y}, \underline{Z}]/I_P$ where $I_P = I_R^e + I_S^e + \langle \underline{Y} \cdot \underline{Z} \rangle$. In particular, if R and S are standard graded, then so is P and, for $i \geq 1$, $H_P(i) = H_R(i) + H_S(i)$.

b) Now suppose R and S are Gorenstein and $Q = R \#_{\mathbf{k}} S$ is their connected sum over \mathbf{k} . Then, by Definition 9, there exist $\Delta_R \in \mathbf{k}[\underline{Y}]$ and $\Delta_S \in \mathbf{k}[\underline{Z}]$ such that their respective images $\delta_R \in R$ and $\delta_S \in S$ generate the respective socles and $Q \simeq (R \times_{\mathbf{k}} S)/\langle \delta_R - \delta_S \rangle \simeq \mathbf{k}[\underline{Y}, \underline{Z}]/I_Q$, where $I_Q = I_P + \langle \Delta_R - \Delta_S \rangle = I_R^e + I_S^e + \langle \underline{Y} \cdot \underline{Z} \rangle + \langle \Delta_R - \Delta_S \rangle$. In particular, if R and S are standard graded, then Q is standard graded if and only if $\text{ll}(R) = \text{ll}(S)$.

A different point of view to study Gorenstein Artin \mathbf{k} -algebras is via Macaulay's inverse systems. The next remark tells us what connected sums are in this light.

Remark 15 (Connected Sums and Inverse Systems). Let \mathbf{k} be a field of characteristic zero. One can study Gorenstein Artin \mathbf{k} -algebras via Macaulay's inverse systems. In this correspondence, such rings correspond to polynomials. For example, the ring $\mathbf{k}[X_1, \dots, X_n]/I$, where $I = \langle X_i X_j, X_i^2 - X_j^2 : 1 \leq i < j \leq n \rangle$ corresponds to the polynomial $Z_1^2 + \dots + Z_n^2$. For more details, see [1, Section 1.4] or [7, Section 2].

It can be seen that if R and S are Gorenstein Artin \mathbf{k} -algebras corresponding to polynomials $F(\underline{Y})$ and $G(\underline{Z})$ respectively, then the Gorenstein Artin \mathbf{k} -algebra corresponding to $F + G$ is a connected sum of R and S over \mathbf{k} . (For example, see [1, Remark 4.24]).

In particular, whenever the polynomial corresponding to a Gorenstein Artin \mathbf{k} -algebra Q is of the form $F(\underline{Y}) + Z_1^2 + \dots + Z_n^2$, then Q is a connected sum, over \mathbf{k} , of the Gorenstein Artin rings corresponding to the polynomials $F(\underline{Y})$ and $Z_1^2 + \dots + Z_n^2$, say R and S respectively. As seen above, S is a Gorenstein Artin local ring with $\text{ll}(S) = 2$. This observation plays an important role in Section 5.

The main question we would like to address is:

Main Question: When is a given Gorenstein Artin local ring Q decomposable as a connected sum over \mathbf{k} ?

Remark 16. *In terms of the inverse systems, answering the main question amounts to the following: Given a polynomial F corresponding to Q , write $F = F_1 + F_2$, where F_1 and F_2 are polynomials in a disjoint set of variables.*

When F is homogeneous, the above property has been studied by the authors in [4]. They define such a polynomial to be a direct sum and the corresponding Gorenstein algebra to be apolar.

Before we proceed to answer the main question, observe the following trivial cases:

Remark 17 (Trivial Fibre Products and Connected Sums). *Let $(R, \mathfrak{m}_R, \mathbf{k})$ be an Artinian local ring different from \mathbf{k} and $S = \mathbf{k}$. Then it is easy to see that $R \times_{\mathbf{k}} S \simeq R$. We call this the trivial fibre product. Note that $R \times_{\mathbf{k}} S$ is a non-trivial fibre product if and only if $\text{ll}(R)$ and $\text{ll}(S)$ are both at least 1.*

Similarly, let $(R, \mathfrak{m}_R, \mathbf{k})$ be a Gorenstein Artin local ring different from \mathbf{k} and S be a ring of length two. Note that this forces S to be Gorenstein. One can check that $R \#_{\mathbf{k}} S \simeq R$. This is referred to as the trivial connected sum decomposition of R . Thus $R \#_{\mathbf{k}} S$ is not a trivial connected sum if and only if $\text{ll}(R), \text{ll}(S) \geq 2$. (A simple length count proves the converse).

In light of this, we make the following key definition about connected sums. Note that a similar terminology is also used for fibre products in this article.

Definition 18. *Let Q be a Gorenstein Artin local \mathbf{k} -algebra. We say that Q can be decomposed non-trivially as a connected sum over \mathbf{k} if there exist Gorenstein Artin \mathbf{k} -algebras R and S such that $Q \simeq R \#_{\mathbf{k}} S$ and $R \not\simeq Q \not\simeq S$.*

If this happens, we say that R and S are the components in a non-trivial connected sum decomposition of Q .

If this does not happen, we say that Q is indecomposable as a connected sum over \mathbf{k} .

It is known (see [2, Theorem 8.3]) that if Q is a *complete intersection* with $\text{edim}(Q) \geq 3$, then Q is indecomposable as a connected sum over \mathbf{k} .

In this section, we give two other necessary conditions: one concerning the second Hilbert coefficient $H_Q(2)$ of Q , and the second in terms of the associated graded ring $\text{gr}(Q)$ of Q . Finally, we give a characterization of connected sums of \mathbf{k} -algebras over \mathbf{k} in terms of the defining ideals. The notation in the rest of this section is as in Remark 14.

3.1. Hilbert Functions. We first obtain a numerical criterion that is satisfied by connected sums.

Proposition 19. *Let R and S be Gorenstein Artin \mathbf{k} -algebras, with $\text{edim}(R) = m$ and $\text{edim}(S) = n$. If $Q \simeq R \#_{\mathbf{k}} S$ is a non-trivial connected sum, then $H_Q(2) \leq \binom{m+n+1}{2} - mn$.*

Proof. With notation as in Remark 14, since $\underline{Y} \cdot \underline{Z} \in I_Q$, we have $Y_i^* Z_j^* \in I_Q^*$ for all i and j . Hence, $H_{\mathbf{k}[\underline{Y}, \underline{Z}]}(2) = \binom{m+n+1}{2}$ implies that

$$H_Q(2) = H_{\text{gr}(Q)}(2) = \binom{m+n+1}{2} - \dim_{\mathbf{k}}((I_Q^* + \langle \underline{Y} \cdot \underline{Z} \rangle^3) / \langle \underline{Y} \cdot \underline{Z} \rangle^3) \leq \binom{m+n+1}{2} - mn. \quad \square$$

Note that for positive integers m and n , if $d = m + n$ is fixed, the minimum value of $mn = m(d - m)$ is obtained when $m = 1$ or $m = d - 1$. Hence, an immediate corollary to the above proposition is:

Theorem 20. *Let Q be a Gorenstein Artin local \mathbf{k} -algebra with $\text{edim}(Q) = d$. If $H_Q(2) \geq \binom{d}{2} + 2$, then Q is indecomposable as a connected sum over \mathbf{k} .*

We say Q is a *compressed Gorenstein \mathbf{k} -algebra* if it has a maximum possible Hilbert function given the embedding dimension d and Loewy length s , i.e., if the Hilbert function of Q is $H_Q(i) = \min\{\binom{d+i-1}{i}, \binom{d+s-i-1}{s-i}\}$.

Corollary 21. *If Q is a compressed Gorenstein \mathbf{k} -algebra with $\text{ll}(Q) \geq 4$, then Q is indecomposable as a connected sum over \mathbf{k} .*

Remark 22. *Since a generic Gorenstein \mathbf{k} -algebra is compressed (see [8]), we see that they are indecomposable as a connected sum over \mathbf{k} . This has also been proved in [15, Proposition 4.4], which the authors obtain by using different techniques.*

3.2. Associated Graded Rings. We first prove a basic property of the associated graded ring of a connected sum of \mathbf{k} -algebras to obtain a necessary condition. We need the following:

Lemma 23. *If both R and S are Artinian \mathbf{k} -algebras, then*

$$\text{gr}(R \times_{\mathbf{k}} S) \simeq \text{gr}(R) \times_{\mathbf{k}} \text{gr}(S).$$

Proof. Let $P = R \times_{\mathbf{k}} S$. By Remark 8(a), we have $R \simeq P/J$, $S \simeq P/I$, and $\mathbf{k} = P/(I + J)$. Thus if $I = \langle \underline{y} \rangle$ and $J = \langle \underline{z} \rangle$, we see that $\mathfrak{m}_P = \langle \underline{y}, \underline{z} \rangle$ is the maximal ideal of P . Hence $\mathfrak{m}_P^* = \langle y_1^*, \dots, y_m^*, z_1^*, \dots, z_n^* \rangle$ is the maximal ideal P_+ of $\text{gr}(P)$. Thus $\langle y_1^*, \dots, y_m^* \rangle \subseteq I^*$ and $\langle z_1^*, \dots, z_n^* \rangle \subseteq J^*$ force $I^* + J^* = P_+$. Since $\lambda(P) = \lambda(R) + \lambda(S) - 1$ and $\lambda(\text{gr}(P)) = \lambda(P)$, we have $\lambda(\text{gr}(P)) = \lambda(\text{gr}(R) \times_{\mathbf{k}} \text{gr}(S))$.

Now, by Remark 2(b), $R = P/J$ and $S = P/I$ imply that $\text{gr}(R) \simeq \text{gr}(P)/J^*$ and $\text{gr}(S) \simeq \text{gr}(P)/I^*$, respectively. In particular, the natural projection $\text{gr}(P) \twoheadrightarrow \mathbf{k}$ factors through the surjective maps $\text{gr}(P) \twoheadrightarrow \text{gr}(R)$ and $\text{gr}(P) \twoheadrightarrow \text{gr}(S)$. Hence $\text{gr}(P)$ maps onto $\text{gr}(R) \times_{\mathbf{k}} \text{gr}(S)$. Since $\lambda(\text{gr}(P)) = \lambda(\text{gr}(R) \times_{\mathbf{k}} \text{gr}(S))$, we get the desired isomorphism. \square

Proposition 24. *Let R and S be Gorenstein Artin \mathbf{k} -algebras with $\text{ll}(R) \neq \text{ll}(S)$. Then the associated ring of $R \#_{\mathbf{k}} S$ is a fibre product.*

Moreover, if $\text{ll}(R)$ and $\text{ll}(S)$ are at least two, the non-trivial connected sum $R \#_{\mathbf{k}} S$ is not a standard graded \mathbf{k} -algebra.

Proof. Let $P = R \times_{\mathbf{k}} S$ and $Q = R \#_{\mathbf{k}} S$. Let $\text{soc}(R) = \langle \delta_R \rangle$ and $\text{soc}(S) = \langle \delta_S \rangle$. Since $Q \simeq P/\langle \delta_R - u\delta_S \rangle$ for some unit u in S , by Remark 2(b), we have $\text{gr}(Q) \simeq \text{gr}(P)/\langle \delta_R - u\delta_S \rangle^*$. Without loss of generality, we may assume that $\text{ll}(R) > \text{ll}(S)$. Hence $\langle \delta_R - u\delta_S \rangle^* = \langle u\delta_S \rangle^*$. Thus we see that $\text{gr}(Q) \simeq (\text{gr}(R) \times_{\mathbf{k}} \text{gr}(S))/\langle u\delta_S \rangle^* \simeq \text{gr}(R) \times_{\mathbf{k}} \text{gr}(S/\langle \delta_S \rangle)$.

Finally, if $\text{ll}(R) > \text{ll}(S) \geq 2$, then $\text{gr}(R) \neq \mathbf{k} \neq \text{gr}(S/\langle \delta_S \rangle)$. Hence $\text{gr}(Q)$ is not Gorenstein, by Remark 8(b). Thus $Q \not\simeq \text{gr}(Q)$, hence Q is not standard graded. \square

The condition $\text{ll}(R) \neq \text{ll}(S)$ is a necessary assumption in the above proposition. Firstly, note that as observed in Remark 14, if R and S are graded with $\text{ll}(R) = \text{ll}(S) \geq 2$, then $Q \simeq R \#_{\mathbf{k}} S$ is graded. In particular, Q is non-trivial connected sum, but $Q \simeq \text{gr}(Q)$ is indecomposable as a fibre product over \mathbf{k} .

The graded nature of R and S does not play a role in $\text{gr}(Q)$ being indecomposable as a fibre product over \mathbf{k} , as can be seen in the following example.

Example 25. *Let $R \simeq \mathbf{k}[Y_1, Y_2]/\langle Y_1^2 Y_2, Y_1^3 - Y_2^2 \rangle$, $S = \mathbf{k}[Z]/\langle Z^5 \rangle$ and $Q \simeq R \#_{\mathbf{k}} S$. Then $G = \text{gr}(Q) \simeq \mathbf{k}[Y_1, Y_2, Z]/\langle Y_1 Z, Y_2 Z, Y_1^2 Y_2, Y_2^2, Y_1^4 - Z^4 \rangle$. Note that in this case, R is not*

standard graded, and $\text{ll}(R) = \text{ll}(S) = 4$. We now show that G is indecomposable as a fibre product over \mathbf{k} .

Suppose $G \simeq A \times_{\mathbf{k}} B$ is a non-trivial fibre product, for some \mathbf{k} -algebras A and B . Since $3 = \text{edim}(G) = \text{edim}(A) + \text{edim}(B)$, we may assume that $\text{edim}(A) = 2$ and $\text{edim}(B) = 1$. Furthermore, $2 = \text{type}(G) = \text{type}(A) + \text{type}(B)$ forces A and B to be Gorenstein, and $\text{soc}(G) = \text{soc}(A) \oplus \text{soc}(B)$ implies that $\text{ll}(A) = 4$ and $\text{ll}(B) = 2$ or vice versa. Since the Hilbert function of B , $H_B(i) = 1$ for each $i \leq \text{ll}(B)$, one can check that if $\text{ll}(B) = 4$, then $\text{ll}(A) = 3$. As this cannot happen, we must have $\text{ll}(A) = 4$ and $\text{ll}(B) = 2$.

Thus, if $G \simeq A \times_{\mathbf{k}} B$ is a non-trivial fibre product, we may assume that A and B are Gorenstein, $\text{edim}(A) = 2$, $\text{ll}(A) = 4$, and $B \simeq \mathbf{k}[V]/\langle V^3 \rangle$. In particular, we can write $A \simeq \mathbf{k}[U_1, U_2]/\langle f_1, f_2 \rangle$. Note that U_1, U_2 and V are indeterminates over \mathbf{k} .

Thus $G \simeq \mathbf{k}[U_1, U_2, V]/\langle U_1V, U_2V, V^3, f_1, f_2 \rangle$. Let small letters denote the respective images of the indeterminates in G and

$$\phi : \mathbf{k}[U_1, U_2, V]/\langle U_1V, U_2V, V^3, f_1, f_2 \rangle \longrightarrow \mathbf{k}[Y_1, Y_2, Z]/\langle Y_1Z, Y_2Z, Y_1^2Y_2, Y_2^2, Y_1^4 - Z^4 \rangle$$

be an isomorphism.

Write $\phi(u_1) = a_{11}y_1 + a_{12}y_2 + a_{13}z$, $\phi(u_2) = a_{21}y_1 + a_{22}y_2 + a_{23}z$ and $\phi(v) = a_{31}y_1 + a_{32}y_2 + a_{33}z$. The fact that $\phi(v^3) = 0$ forces $a_{31} = 0 = a_{33}$. Hence $a_{32} \neq 0$ and therefore, $a_{11} = 0 = a_{21}$ since $\phi(u_1v) = 0 = \phi(u_2v)$. This gives us a contradiction as $a_{i1} = 0$ for all i implies that $y_1 \notin \text{im}(\phi)$. Hence G cannot be written as a non-trivial fibre product.

Proposition 24 gives us a strategy to answer the main question, one may expect that if the associated graded ring is indecomposable as a fibre product over \mathbf{k} , then the Gorenstein ring itself is indecomposable as a connected sum over \mathbf{k} . However, the above example shows that this is not necessarily true. Hence, the best possible answer to the main question using this approach is the following application of Proposition 24:

Theorem 26. *Let Q be a Gorenstein Artin \mathbf{k} -algebra such that its associated graded ring $\text{gr}(Q)$ is indecomposable as a fibre product over \mathbf{k} . If R and S are Gorenstein Artin \mathbf{k} -algebras such that $Q \simeq R \#_{\mathbf{k}} S$ is a non-trivial connected sum over \mathbf{k} , then:*

- a) $\text{ll}(R) = \text{ll}(S)$ and
- b) if Q is not graded, then at least one of R and S is not a graded Gorenstein ring.

Proof. Suppose $Q \simeq R \#_{\mathbf{k}} S$ is a non-trivial connected sum over \mathbf{k} . By Proposition 24, since $\text{gr}(Q)$ is indecomposable as a fibre product over \mathbf{k} , we must have $\text{ll}(R) = \text{ll}(S)$, proving (a).

If both R and S are graded, then $\text{ll}(R) = \text{ll}(S)$ forces Q to be graded, which contradicts the hypothesis in (b). Hence (b) holds. \square

3.3. A Characterization in terms of the Defining Ideals. The following theorem gives us a criteria to determine whether a given Gorenstein Artin local \mathbf{k} -algebra is a connected sum.

Proposition 27. *Let $Q = \mathbf{k}[\underline{Y}, \underline{Z}]/I_Q$ be a Gorenstein Artin local ring. Let $R = \mathbf{k}[\underline{Y}]/I_R$ and $S = \mathbf{k}[\underline{Z}]/I_S$, where $I_R = I_Q \cap \mathbf{k}[\underline{Y}]$ and $I_S = I_Q \cap \mathbf{k}[\underline{Z}]$. Suppose $Y_i \cdot Z_j \in I_Q$ for $1 \leq i \leq m$, $1 \leq j \leq n$. Then*

- a) R and S are Gorenstein Artin and
- b) $Q \simeq R \#_{\mathbf{k}} S$.

Proof. The inclusions $\mathbf{k}[\underline{Y}], \mathbf{k}[\underline{Z}] \hookrightarrow \mathbf{k}[\underline{Y}, \underline{Z}]$ induce inclusions $R \hookrightarrow Q$ and $S \hookrightarrow Q$. Let y and z denote the respective images of Y and Z in the quotient rings Q, R and S .

a) Let $f \in \text{soc}(R)$. Then $Y_i \cdot F \in I_R \subseteq I_Q$ for each i , where $F \in \mathbf{k}[\underline{Y}]$ is a preimage in $\mathbf{k}[\underline{Y}]$ of f . Moreover, since $Y_i Z_j \in I_Q$ for each i and j , $Z_j F \in I_Q$. Hence $f \in \text{soc}(Q)$. Therefore $0 \neq \text{soc}(R) \subseteq \text{soc}(Q)$ which is a one-dimensional \mathbf{k} -vector space. Thus $\dim_{\mathbf{k}}(\text{soc}(R)) = 1$, i.e., R is Gorenstein Artin.

We can show that S is also a Gorenstein Artin local ring by a similar argument.

b) Let $P = R \times_{\mathbf{k}} S$. Then $P \simeq \mathbf{k}[\underline{Y}, \underline{Z}]/I_P$, where $I_P = I_R^e + I_S^e + \langle \underline{Y} \cdot \underline{Z} \rangle$. By the hypothesis, $I_P \subseteq I_Q$ and hence induces a natural surjective map $\pi : P \rightarrow Q$.

Let δ_R and δ_S be the generators of $\text{soc}(R)$ and $\text{soc}(S)$ respectively, with their respective preimages in $\mathbf{k}[\underline{Y}, \underline{Z}]$ being Δ_R and Δ_S . Note that if $\Delta_R \in I_Q$, then $\Delta_R \in I_R$, which is not true. Similarly, $\Delta_S \notin I_Q$, i.e., $\pi(\delta_R) \neq 0 \neq \pi(\delta_S)$ in Q . But $\text{soc}(P) = \langle \delta_R, \delta_S \rangle$ maps into $\text{soc}(Q)$, which is one-dimensional. Hence $\pi(\delta_R) = u\pi(\delta_S)$ for some unit $u \in \mathbf{k}$. Thus $\delta_R - u\delta_S \in \ker(\pi)$ for some $u \in \mathbf{k}$.

Finally, we show that $\ker(\pi) \subseteq \text{soc}(P)$. Let $f \in \ker(\pi)$ and F be a preimage in $\mathbf{k}[\underline{Y}, \underline{Z}]$. In order to prove $f \in \text{soc}(P)$, it is enough to prove that $Y_i F, Z_j F \in I_P$ for all i and j .

We show that $Y_1 F \in I_P$. The others follow similarly. Write $F = F_1(\underline{Y}) + F_2(\underline{Y}, \underline{Z})$, where every term of F_2 is a multiple of some Z_j . Then $Y_1 F_2 \in I_P \subseteq I_Q$.

Now $f \in \ker(\pi)$ implies $F \in I_Q$. Thus $Y_1 F_1 = Y_1 F - Y_1 F_2 \in I_Q \cap \mathbf{k}[\underline{Y}] = I_R$. Therefore, $Y_1 F_1$ is also in I_P , which implies that $Y_1 F \in I_P$. This proves that $\ker(\pi) \subseteq \text{soc}(P)$.

Thus, we have

$$0 \subsetneq \langle \delta_R - u\delta_S \rangle \subseteq \ker(\pi) \subsetneq \langle \delta_R, \delta_S \rangle = \text{soc}(P).$$

Since $\lambda(\text{soc}(P)) = 2$, this forces $\ker(\pi) = \langle \delta_R - u\delta_S \rangle$. Therefore, $Q \simeq P/\langle \delta_R - u\delta_S \rangle$ for some unit $u \in \mathbf{k}$, and hence is a connected sum of R and S over \mathbf{k} . \square

The requirement that $\underline{Y} \cdot \underline{Z} \in I_Q$ is necessary in the above proposition, otherwise R or S may not be Gorenstein as can be seen from the following example.

Example 28. Let $Q = \mathbf{k}[Y_1, Z_1, Z_2]/I_Q$, where $I_Q = \langle Y_1 Z_1 - Z_2^2, Y_1^2, Z_1^2 \rangle$. Then $I_Q \cap \mathbf{k}[Z_1, Z_2] = \langle Z_1^2, Z_1 Z_2^2, Z_2^4 \rangle$. (One can verify this using a computer algebra package. For example, we use the elimination package in Macaulay2). Note that Q is Gorenstein, but $S = \mathbf{k}[Z_1, Z_2]/(I_Q \cap \mathbf{k}[Z_1, Z_2])$ is not Gorenstein because it has a two dimensional socle.

It follows from Remark 8(a) that, if $P = R \times_{\mathbf{k}} S$, then we can identify R and S with appropriate quotients of P . On the other hand, if $Q = R \#_{\mathbf{k}} S$, in general, it is not clear how one can recover the components R and S from Q . Proposition 29 shows that R and S can be identified with subrings in the case of \mathbf{k} -algebras.

Proposition 29. Let P, Q, R and S be \mathbf{k} -algebras with $P \simeq \mathbf{k}[\underline{Y}, \underline{Z}]/I_P$, $Q \simeq \mathbf{k}[\underline{Y}, \underline{Z}]/I_Q$, $R \simeq \mathbf{k}[\underline{Y}]/I_R$ and $S \simeq \mathbf{k}[\underline{Z}]/I_S$.

a) If $P \simeq R \times_{\mathbf{k}} S$, then $I_R = I_P \cap \mathbf{k}[\underline{Y}]$ and $I_S = I_P \cap \mathbf{k}[\underline{Z}]$.

b) If R and S are Gorenstein Artin with $Q \simeq R \#_{\mathbf{k}} S$, then $I_R = I_Q \cap \mathbf{k}[\underline{Y}]$ and $I_S = I_Q \cap \mathbf{k}[\underline{Z}]$.

Proof.

a) By Remark 14(a), since $I_P = I_R^e + I_S^e + \langle \underline{Y} \cdot \underline{Z} \rangle$, it is enough to show $I_P \cap \mathbf{k}[\underline{Y}] \subseteq I_R$.

Let $F(\underline{Y}) \in I_P \cap \mathbf{k}[\underline{Y}]$. We can write $F = F_1(\underline{Y}) + F_2(\underline{Y}, \underline{Z})$, where $F_1 \in I_R$ and every term of F_2 is a multiple of some Z_j . But $F - F_1 \in \mathbf{k}[\underline{Y}]$, hence $F_2 = F - F_1 = 0$.

b) Let $I_{R'} = I_Q \cap \mathbf{k}[\underline{Y}]$ and $I_{S'} = I_Q \cap \mathbf{k}[\underline{Z}]$. Since $I_R^e + I_S^e \subseteq I_Q$, we have $I_R \subseteq I_{R'}$ and $I_S \subseteq I_{S'}$, which induce natural surjective maps $\pi_1 : R \rightarrow R'$ and $\pi_2 : S \rightarrow S'$, where $R' = \mathbf{k}[\underline{Y}]/I_{R'}$, $S' = \mathbf{k}[\underline{Z}]/I_{S'}$. In particular, $\lambda(R') \leq \lambda(R)$ and $\lambda(S') \leq \lambda(S)$. In order to

prove that $I_R = I_{R'}$ and $I_S = I_{S'}$, it is enough to prove that π_1 and π_2 are isomorphisms, in particular, it is enough to show that $\lambda(R') = \lambda(R)$ and $\lambda(S') = \lambda(S)$.

Since $Y_i \cdot Z_j \in I_Q$ for $1 \leq i \leq m$, $1 \leq j \leq n$, by Theorem 27, R' and S' are Gorenstein Artin and $Q \simeq R' \#_k S'$. Hence $\lambda(R') + \lambda(S') = \lambda(Q) + 2 = \lambda(R) + \lambda(S)$. Since $\lambda(R') \leq \lambda(R)$ and $\lambda(S') \leq \lambda(S)$, we get $\lambda(R') = \lambda(R)$ and $\lambda(S') = \lambda(S)$, completing the proof. \square

We conclude this section with a result on the minimal number of generators of the defining ideals of P and Q .

Proposition 30. *With notation as in Remark 14, we have the following:*

- a) $\mu(I_P) = \mu(I_R) + \mu(I_S) + mn$.
- b) $\mu(I_Q) = \mu(I_R) + \mu(I_S) + mn + \psi = \mu(I_P) + \psi$,
where $\psi = 1$ when $m, n \geq 2$, $\psi = -1$ when $m = 1 = n$, and $\psi = 0$ otherwise.

Proof.

a) Comparing the coefficients of t and t^2 in Equation 13.1, we see that $\beta_1^P = \beta_1^R + \beta_1^S$ and $\beta_2^P = \beta_2^R + \beta_2^S + 2\beta_1^R\beta_1^S$. Hence (a) follows by Remark 5(b).

b) First of all, note that if $m = 1 = n$, then $\mu(I_R) = 1 = \mu(I_S)$. Since Q is Gorenstein Artin with $\text{edim}(Q) = m + n = 2$, it is a well known result of Serre that $\mu(I_Q) = 2$. Hence, without loss of generality, we may assume that $m \geq 2$.

It is clear from Equation 13.2 that $\beta_2^Q = \beta_2^P$ when $n = 1$ and $\beta_2^Q = \beta_2^P + 1$ when $n \geq 2$. Since $\beta_1^Q = \beta_1^P$ in either case, the proof of (b) is complete by Remark 5(b). \square

We use Proposition 27 and Proposition 29(b) to give equivalent conditions for a Gorenstein Artin k -algebra to be a connected sum over k . We can summarize the results obtained in this subsection in the following:

Theorem 31. *Suppose R, S , and Q are Artin k -algebras of the form $R \simeq k[\underline{Y}]/I_R$, $S \simeq k[\underline{Z}]/I_S$ and $Q \simeq k[\underline{Y}, \underline{Z}]/I_Q$. Then the following are equivalent:*

- i) R and S are Gorenstein Artin, and $Q = R \#_k S$.
- ii) Q is Gorenstein Artin, $I_R = I_Q \cap k[\underline{Y}]$, $I_S = I_Q \cap k[\underline{Z}]$, and $Y_i Z_j \in I_Q$ for all i, j .

If the above conditions are satisfied, then $\mathbb{P}^Q(t)$ is given by Equation (13.2), $\mu(I_Q)$ is given by the formula in Proposition 30(b) and $I_Q = I_R^e + I_S^e + \langle \underline{Y} \cdot \underline{Z} \rangle + \langle \Delta_R - \Delta_S \rangle$, where $\Delta_R \in k[\underline{Y}]$ and $\Delta_S \in k[\underline{Z}]$ are such that their respective images $\delta_R \in R$ and $\delta_S \in S$ generate the respective socles.

Proof. Note that (ii) implies (i) by Proposition 27. If (i) holds, we first observe that Q is Gorenstein by Theorem 12. The rest of the theorem follows from Proposition 29(b), Remark 13(b), Proposition 30(b) and Remark 14(b). \square

4. ASSOCIATED GRADED RINGS AND CONNECTED SUMS

In this section, we explore the connections between associated graded rings and connected sums. In particular, we study conditions on the associated graded ring of a Gorenstein Artin k -algebra which force it to be a connected sum. We note that as an immediate consequence of Proposition 24, we get the following:

Corollary 32. *Let R and S be Gorenstein Artin k -algebras where $\text{ll}(S) = 2$ and $\text{ll}(R) \geq 3$. If $Q \simeq R \#_k S$, then there is a surjective map $\pi : G = \text{gr}(Q) \twoheadrightarrow A = \text{gr}(R)$ such that $\ker(\pi)_i = 0$, $i \geq 2$.*

The main theorem we prove in this section (Theorem 39) shows that the converse of Corollary 32 is true when A itself is a graded Gorenstein ring.

With A and G as in Corollary 32, if A is further assumed to be graded Gorenstein, one can see that $\text{soc}(G) \cap (G_+)^2 \simeq \text{soc}(A)$, which is a one-dimensional \mathbf{k} -vector space. In fact, as we show in Proposition 34, these properties are equivalent. Hence, we introduce the following:

Definition 33. *We say that a graded Artinian \mathbf{k} -algebra G is Gorenstein up to linear socle if the socle of G in degree two and higher is a one-dimensional vector space over \mathbf{k} , i.e., $\dim_{\mathbf{k}}(\text{soc}(G) \cap (G_+)^2) = 1$.*

In Theorem 39, we prove that if the associated graded ring of a Gorenstein Artin local ring Q , with Loewy length at least 3, is Gorenstein up to linear socle, then Q can be decomposed as a connected sum.

Before we state the main theorem, we explore what it means to be Gorenstein up to linear socle. We begin with the following proposition which characterizes this property.

Proposition 34. *Let G be a graded \mathbf{k} -algebra with $\text{ll}(G) \geq 2$. Then the following are equivalent:*

- i) G is Gorenstein up to linear socle.
- ii) There exist graded \mathbf{k} -algebras A and B , where A is graded Gorenstein and $B_+^2 = 0$ such that $G \simeq A \times_{\mathbf{k}} B$.
- iii) There exists a graded Gorenstein ring A and a surjective ring homomorphism $\pi : G \twoheadrightarrow A$ such that $\ker(\pi) \cap (G_+)^2 = 0$.

Proof. First of all, note that if G itself is Gorenstein, then the above statements are all trivially true. G satisfies (i), we can take $A = G$ in (iii), and write G as the trivial fibre product $G \times_{\mathbf{k}} \mathbf{k}$ in (ii). Hence, we may assume that G itself is not Gorenstein, i.e., $(\text{soc}(G) + (G_+)^2)/(G_+)^2 \neq 0$.

(i) \Rightarrow (ii): Let G be Gorenstein up to linear socle. Then we can find a \mathbf{k} -basis, say $\{\overline{z_1}, \dots, \overline{z_n}\}$, for $(\text{soc}(G) + (G_+)^2)/(G_+)^2$, where $n = \text{type}(G) - 1$. Extend this to a \mathbf{k} -basis $\{\overline{y_1}, \dots, \overline{y_m}, \overline{z_1}, \dots, \overline{z_n}\}$ of $G_+/(G_+)^2$, and lift it to a minimal generating set $\{\underline{y}, \underline{z}\}$ of G_+ in G_1 . Since $\langle \underline{y} \rangle \cap \langle \underline{z} \rangle = 0$ and $\langle \underline{y} \rangle + \langle \underline{z} \rangle = G_+$, it is easy to see that $G \simeq A \times_{\mathbf{k}} B$, where $A = G/\langle \underline{z} \rangle$ and $B = G/\langle \underline{y} \rangle$ are graded \mathbf{k} -algebras.

Since $z_i \in \text{soc}(G)$ for each i , their images in B , which are the generators of B_+ , are in $\text{soc}(B)$, in particular, $B_+ = \text{soc}(B)$. Thus $B_+^2 = 0$. Furthermore, $\text{type}(B) = n$ and $\text{type}(G) = \text{type}(A) + \text{type}(B)$ implies $\text{type}(A) = 1$, i.e., A is a graded Gorenstein \mathbf{k} -algebra, proving (ii).

(ii) \Rightarrow (iii): Suppose (ii) is true. By Remark 8(a), $A \simeq G/J$, where $J = \{(0, b) : b \in B_+\}$. Since $B_+^2 = 0$, (iii) holds.

(iii) \Rightarrow (i): Assuming (iii), we see that $\pi|_{(G_+)^i} : (G_+)^i \rightarrow (A_+)^i$ is an isomorphism for each $i \geq 2$. In particular, if $\text{soc}(A) = (A_+)^s$, then $s = \text{ll}(A) = \text{ll}(G) \geq 2$ and $\dim_{\mathbf{k}}((G_+)^s) = 1$.

Suppose $z \in \text{soc}(G) \cap (G_+)^2$. Then $\pi(z) \in \text{soc}(A) = (A_+)^s$. Since $s \geq 2$, this forces $z \in (G_+)^s$. Thus $(G_+)^s \subseteq \text{soc}(G) \cap (G_+)^2 \subseteq (G_+)^s$, proving (i). \square

The following remark lists observations from Proposition 34 which we use repeatedly.

Remark 35. *Let (Q, \mathfrak{m}_Q, k) be a Gorenstein Artin local ring such that $G = \text{gr}(Q)$ is Gorenstein up to linear socle. Let the notation be as in Proposition 34.*

Firstly, note that $\text{edim}(Q) = \text{edim}(G) = \text{edim}(A) + \text{edim}(B)$ and $\lambda(Q) = \lambda(G) = \lambda(A) + \lambda(B) - 1$. Hence, if $n = \text{type}(B) = \text{type}(G) - \text{type}(A) = \text{type}(G) - 1$, then since $\text{edim}(B) = \text{type}(B) = \lambda(B) - 1$, we get $\lambda(A) = \lambda(Q) - n$ and $\text{edim}(A) = \text{edim}(Q) - n$.

Furthermore, by the proof of (i) \Rightarrow (ii), we note that $A \simeq G/\langle \text{soc}(G) \cap G_1 \rangle$. Hence, $\text{ll}(A) = \text{ll}(Q)$.

Key Note: With notation as above, note that G itself is Gorenstein if and only if $n = \text{type}(G) - 1 = 0$. For the next two results, we will assume that G is Gorenstein up to linear socle but not Gorenstein, i.e., $\text{type}(G) = n + 1 \geq 2$.

In the following lemma, we record some properties of Gorenstein rings whose associated graded rings are Gorenstein up to linear socle.

Lemma 36. *Let (Q, \mathfrak{m}_Q, k) be a Gorenstein Artin local ring with $\text{ll}(Q) = s$ such that $G = \text{gr}(Q)$ is Gorenstein up to linear socle but not Gorenstein. If $s \geq 3$, then the following hold:*

- a) $(0 :_Q \mathfrak{m}_Q^2) \cap \mathfrak{m}_Q^2 = \mathfrak{m}_Q^{s-1}$.
- b) $(0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}$ is a k -vector space of dimension $\text{type}(G) - 1$.
- c) If $w \in (0 :_Q \mathfrak{m}_Q^2) \setminus \mathfrak{m}_Q^{s-1}$, then $w \in \mathfrak{m}_Q \setminus \mathfrak{m}_Q^2$, $w \cdot \mathfrak{m}_Q = \text{soc}(Q)$ and $w^* \in \text{soc}(G) \setminus (G_+)^2$.

Proof. By Proposition 34, there exists a graded Gorenstein ring A and a surjective ring homomorphism $\pi : G \twoheadrightarrow A$ such that $\ker(\pi) \cap (G_+)^2 = 0$. Note that the induced map $\pi : (G_+)^i \rightarrow (A_+)^i$ is an isomorphism for each $i \geq 2$.

a) Let $w \in (0 :_Q \mathfrak{m}_Q^2)$. Then $\pi(w^*) \in (0 :_A (A_+)^2)$. Since A is graded Gorenstein, we have $(0 :_A (A_+)^2) = (A_+)^{s-1}$ and hence $\pi(w^*) \in (A_+)^{s-1}$. Suppose further $w \in \mathfrak{m}_Q^2$. Then $\deg(w^*) \geq 2$ in G . Since $\pi : G_i \rightarrow A_i$ is an isomorphism for $i \geq 2$, $\pi(w^*) \in (A_+)^{s-1}$ forces $w^* \in (G_+)^{s-1}$, i.e., $w \in \mathfrak{m}_Q^{s-1}$. Thus $(0 :_Q \mathfrak{m}_Q^2) \cap \mathfrak{m}_Q^2 \subseteq \mathfrak{m}_Q^{s-1}$. The other inclusion is clear since $\mathfrak{m}_Q^s = \text{soc}(Q)$.

b) By (a),

$$\frac{(0 :_Q \mathfrak{m}_Q^2)}{\mathfrak{m}_Q^{s-1}} \simeq \frac{(0 :_Q \mathfrak{m}_Q^2)}{(0 :_Q \mathfrak{m}_Q^2) \cap \mathfrak{m}_Q^2} \simeq \frac{(0 :_Q \mathfrak{m}_Q^2) + \mathfrak{m}_Q^2}{\mathfrak{m}_Q^2}$$

which is a k -vector space since $\mathfrak{m}_Q^2 \neq 0$.

Let $n = \lambda(\ker(\pi)) = \text{edim}(Q) - \text{edim}(A)$. Note that since $(0 :_Q \mathfrak{m}_Q^2)$ is the canonical module of Q/\mathfrak{m}_Q^2 , $\lambda(0 :_Q \mathfrak{m}_Q^2) = \lambda(Q/\mathfrak{m}_Q^2)$. Also, $\ker(\pi)_i = 0$ for $i \geq 2$ gives $\lambda(\mathfrak{m}_Q^i) = \lambda((G_+)^i) = \lambda((A_+)^i)$ for $i \geq 2$. Hence $\lambda((0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}) = \lambda(Q/\mathfrak{m}_Q^2) - \lambda((A_+)^{s-1}) = \lambda(Q/\mathfrak{m}_Q^2) - \lambda(A/(A_+)^2)$, where the last equality follows from $\lambda((A_+)^{s-1}) = \lambda(A/(A_+)^2)$, which holds since A is a graded Gorenstein ring.

Thus $\dim_k((0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}) = \lambda((0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}) = \text{edim}(Q) - \text{edim}(A) = \text{type}(G) - 1 = n$ by Remark 35.

c) If $w \in \mathfrak{m}_Q^2$, then $w \in \mathfrak{m}_Q^2 \cap (0 :_Q \mathfrak{m}_Q^2) = \mathfrak{m}_Q^{s-1}$ by (a). Hence $w \notin \mathfrak{m}_Q^2$ and so $w^* \notin (G_+)^2$. Now $w \cdot \mathfrak{m}_Q^2 = 0$ implies that $w\mathfrak{m}_Q \subseteq \text{soc}(Q)$. Since Q is Gorenstein and $s \geq 3$, $\text{soc}(Q) \subseteq w\mathfrak{m}_Q$ proving $w\mathfrak{m}_Q = \text{soc}(Q) = \mathfrak{m}_Q^s$. In particular, since $s \geq 3$, $w^* \in \text{soc}(G) \setminus (G_+)^2$. \square

We now prove the following proposition, which is crucial in our proof of the main theorem.

Proposition 37. *Let (Q, \mathfrak{m}_Q, k) be a Gorenstein Artin local ring such that $G = \text{gr}(Q)$ is Gorenstein up to linear socle but not Gorenstein. If $\text{ll}(Q) \geq 3$, there is an ideal J in Q such that the following hold:*

- a) $J\mathfrak{m}_Q^2 = 0$, and hence $J\mathfrak{m}_Q \subseteq \text{soc}(Q)$,
- b) $\mu(J) = \text{type}(G) - 1$,
- c) $0 :_Q J + J = \mathfrak{m}_Q$,
- d) $\mathfrak{m}_Q^r = (0 :_Q J)^r$ for $r \geq 2$ and
- e) $\mu(0 :_Q J) + \mu(J) = \text{edim}(Q)$, and hence $J^2 \neq 0$.

Proof. By Lemma 36(b), there exist elements $z_1, \dots, z_n \in (0 :_Q \mathfrak{m}_Q^2) \setminus \mathfrak{m}_Q^{s-1}$, where $n = \text{type}(G) - 1$, such that their images form a basis for $(0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}$. By Lemma 36(c), $\underline{z} \in \mathfrak{m}_Q \setminus \mathfrak{m}_Q^2$. Now (a) follows by setting $J = \langle z_1, \dots, z_n \rangle$.

Notice that \underline{z} is a part of a minimal generating set of \mathfrak{m}_Q , i.e., \underline{z} are linearly independent modulo \mathfrak{m}_Q^2 . Indeed, suppose $\sum_{i=1}^n a_i z_i \in \mathfrak{m}_Q^2$. Then $\sum_{i=1}^n a_i z_i \in \mathfrak{m}_Q^{s-1}$ by Lemma 36(a), and hence $\sum a_i \bar{z}_i = 0$ in $(0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}$. Since $\{\bar{z}_1, \dots, \bar{z}_n\}$ is linearly independent, $a_i \in \mathfrak{m}_Q$ for all i . In particular, \underline{z} forms a minimal generating set for J showing that $\mu(J) = \text{type}(G) - 1$, which proves (b).

Let $I = 0 :_Q J$. If $w \in J \subseteq (0 :_Q \mathfrak{m}_Q^2)$, then $w \cdot \mathfrak{m}_Q^{s-1} = 0$. Hence, if $w \in J \cap I$, then $w \in (0 :_Q (J + \mathfrak{m}_Q^{s-1})) = (0 :_Q (0 :_Q \mathfrak{m}_Q^2)) = \mathfrak{m}_Q^2$ since Q is Gorenstein Artin. Thus $w \in \mathfrak{m}_Q^{s-1}$ by Lemma 36(a). Write $w = \sum_{i=1}^n a_i z_i$. Since $w \in \mathfrak{m}_Q^{s-1}$ and \underline{z} are linearly independent modulo \mathfrak{m}_Q^{s-1} , $a_i \in \mathfrak{m}_Q$. Thus, $w \in J \cdot \mathfrak{m}_Q \subseteq \text{soc}(Q)$. Therefore $J \cap I \subseteq \text{soc}(Q)$. Since Q is Gorenstein and $J \neq 0 \neq I$, the other inclusion is clear, giving $J \cap I = \text{soc}(Q)$. Finally, since $0 :_Q I = (0 :_Q (0 :_Q J)) = J$, we see that $I + J = \mathfrak{m}_Q$ by taking annihilators of both sides in Q , proving (c).

In order to prove (d), let A and $\pi : G \rightarrow A$ be as in Proposition 34. Since $z_i \in 0 :_Q \mathfrak{m}_Q^2$, $\pi(z_i^*) \in 0 :_A (A_+)^2 = (A_+)^{s-1}$. But $z_i \in \mathfrak{m}_Q \setminus \mathfrak{m}_Q^2$ implies that $\deg(z_i^*) = 1$ in G . Hence either $\deg(\pi(z_i^*)) = 1$ or $\pi(z_i^*) = 0$ in A . Since $\pi(z_i^*) \in (A_+)^{s-1}$, $\deg(\pi(z_i^*)) \neq 1$, forcing $\pi(z_i^*) = 0$ in A . Counting lengths, we see that $\ker(\pi) = \langle z_1^*, \dots, z_n^* \rangle$.

Let $\{y_1, \dots, y_m\}$ be a minimal generating set for I . Since $A_+ = \pi(G_+)$ and $\pi(z_i^*) = 0$, we get $A_+ = \langle \pi(y_1^*), \dots, \pi(y_m^*) \rangle$. Thus $(A_+)^r = \langle \pi(y_1^*), \dots, \pi(y_m^*) \rangle^r$ for each r . Therefore, the fact that $\pi : (G_+)^r \rightarrow (A_+)^r$ is an isomorphism for $r \geq 2$ forces $(G_+)^r = \langle \underline{y}^* \rangle^r$ and hence $\mathfrak{m}_Q^r = I^r$ for $r \geq 2$.

Finally, observe that $J^2 = 0$, implies $J \subseteq 0 :_Q J$ and hence $0 :_Q J = \mathfrak{m}_Q$. Now, $\mu(J) = n \neq 0$ by assumption. Hence, $J^2 \neq 0$ follows if we prove $\mu(I) + \mu(J) = \mu(\mathfrak{m}_Q) = \text{edim}(Q)$.

Now $J\mathfrak{m}_Q = \text{soc}(Q)$ implies that $\lambda(J) = \lambda(J/J\mathfrak{m}_Q) + 1 = n + 1$, and $I^2 \subseteq I\mathfrak{m}_Q \subseteq \mathfrak{m}_Q^2 = I^2$ forces $I\mathfrak{m}_Q = \mathfrak{m}_Q^2$. Since $I \simeq 0 :_Q J$ is isomorphic to the canonical module of Q/J , we see that

$$\begin{aligned} \mu(I) &= \lambda(I/\mathfrak{m}_Q^2) = \lambda(I) - \lambda(\mathfrak{m}_Q^2) = \lambda(Q/J) - \lambda(\mathfrak{m}_Q^2) = \lambda(Q/\mathfrak{m}_Q^2) - \lambda(J) \\ &= 1 + \lambda(\mathfrak{m}_Q/\mathfrak{m}_Q^2) - (n + 1) = \text{edim}(Q) - n = \text{edim}(A), \end{aligned}$$

which proves (e). □

Remark 38. The following is an important observation which is hidden in the proof of (d) above: With hypothesis and notation as in Proposition 37, $\langle \text{soc}(G) \cap G_1 \rangle = \langle z_1^*, \dots, z_n^* \rangle$.

We are now ready to state and prove the main theorem of this section.

Theorem 39. Let $(Q, \mathfrak{m}_Q, \mathbf{k})$ be a Gorenstein Artin \mathbf{k} -algebra such that $G = \text{gr}(Q)$ is Gorenstein up to linear socle. If $\text{ll}(Q) \geq 3$, then there exist Gorenstein Artin local \mathbf{k} -algebras R and S such that $Q \simeq R \#_{\mathbf{k}} S$, where

- a) $\text{ll}(S) \leq 2$ and $\text{gr}(R) \simeq G/\langle \text{soc}(G) \cap G_1 \rangle$.
 b) $\mathbb{P}^Q(t)$ is rational in t if and only if $\mathbb{P}^R(t)$ is so.
 Furthermore, if G is not Gorenstein, then $\text{ll}(S) = 2$.

Proof. If G is Gorenstein, then one can take $Q = R$ and $S = \mathbf{k}[Z]/\langle Z^2 \rangle$. Then the conclusions of the theorem are trivially true. Hence, we now assume that G is not Gorenstein.

By Proposition 37, we see that $\mathbf{m}_Q = \langle y_1, \dots, y_m, z_1, \dots, z_n \rangle$ where $n = \text{type}(G) - 1 \neq 0$, $y \cdot z = 0$ and $\langle z \rangle^2 \neq 0$. Hence we can write $Q \simeq \tilde{Q}/I_Q$ where $\tilde{Q} = \mathbf{k}[Y_1, \dots, Y_m, Z_1, \dots, Z_n]$, $I_Q \subseteq \langle \underline{Y}, \underline{Z} \rangle^2$, $\underline{Y} \cdot \underline{Z} \in I_Q$, $\langle \underline{Z} \rangle^3 \subseteq I_Q$ and $\langle \underline{Z} \rangle^2 \not\subseteq I_Q$.

Hence, if $I_R = I_Q \cap \mathbf{k}[\underline{Y}]$ and $I_S = I_Q \cap \mathbf{k}[\underline{Z}]$, then by Theorem 31, $R = \mathbf{k}[\underline{Y}]/I_R$ and $S = \mathbf{k}[\underline{Z}]/I_S$ are Gorenstein Artin such that $Q \simeq R \#_{\mathbf{k}} S$.

Now, $\langle \underline{Y}, \underline{Z} \rangle^2 \cdot Z_j \subseteq I_Q$ for each j implies $\langle \underline{Z} \rangle^3 \subseteq I_S$ proving $\mathbf{m}_S^3 = 0$. On the other hand, $\langle \underline{Z} \rangle^2 \not\subseteq I_Q$ and hence, $\mathbf{m}_S^2 \neq 0$. Thus $\text{ll}(S) = 2$.

Since $\text{ll}(Q) \geq 3 > \text{ll}(S)$, we see that $\text{ll}(R) = \text{ll}(Q) \neq \text{ll}(S)$. Hence, by (the proof of) Proposition 24, $G \simeq \text{gr}(R) \times_{\mathbf{k}} \text{gr}(S/\text{soc}(S))$. In particular, $\text{gr}(R) \simeq G/\langle z_1^*, \dots, z_n^* \rangle$. By Remark 38, we get $\text{gr}(R) \simeq G/\langle \text{soc}(G) \cap G_1 \rangle$, proving (a).

Finally, note that $\text{ll}(R) = \text{ll}(Q) \geq 3$. Thus, the proof of (b) in this case follows from Remark 6(c) and Remark 13(b). \square

Corollary 40. *Let the hypothesis and notation be as in Theorem 39 and its proof. If we further assume that G is not Gorenstein, then $\mu(I_Q) = \mu(I_R) + \binom{n+1}{2} + mn$ when $m \geq 2$ and $\mu(I_Q) = \binom{n+1}{2} + n$ when $m = 1$.*

Proof. By Theorem 39, since G is not Gorenstein, we have $\text{ll}(S) = 2$. Thus, if $n = \text{edim}(S) \geq 2$, then by Remark 6(c) and Remark 5(b), $\mu(I_S) = \binom{n+1}{2} - 1$. Hence the proof follows from Proposition 30(b). \square

Corollary 41. *Let $(Q, \mathbf{m}_Q, \mathbf{k})$ be a Gorenstein Artin \mathbf{k} -algebra such that $G = \text{gr}(Q)$ is Gorenstein up to linear socle. If $\text{ll}(Q) \geq 3$, $\mathbb{P}^Q(t)$ is a rational function of t in the following situations:*

- i) $\text{char}(\mathbf{k}) \neq 2$ and $\text{edim}(Q) - \text{type}(G) = 3$, or $\text{edim}(Q) - \text{type}(G) \leq 2$.
- ii) $\lambda(Q) - \text{type}(G) \leq 10$.

Proof. Note that by Theorem 39(a), there are Gorenstein Artin local rings R and S such that $Q \simeq R \#_{\mathbf{k}} S$, where $\text{ll}(S) \leq 2$ and $\text{gr}(R) \simeq G/\langle \text{soc}(G) \cap G_1 \rangle$.

Let $A = G/\langle \text{soc}(G) \cap G_1 \rangle$. By Remark 35, A is a graded Gorenstein Artin \mathbf{k} -algebra such that $\text{ll}(A) = \text{ll}(Q)$, $\text{edim}(A) = \text{edim}(Q) - \text{type}(G) + 1$ and $\lambda(A) = \lambda(Q) - \text{type}(G) + 1$.

(ii) The assumption that $\lambda(Q) - \text{type}(G) \leq 10$ implies that $\lambda(A) \leq 11$. Since A is graded Gorenstein, and hence has a palindromic Hilbert function, the hypothesis that $\text{ll}(Q) \geq 3$ forces $\text{edim}(A) \leq 4$. Thus (ii) reduces to (i).

(i) In this case, $\text{char}(\mathbf{k}) \neq 2$ and $\text{edim}(R) = \text{edim}(A) = 4$, or $\text{edim}(R) = \text{edim}(A) \leq 3$. Hence, by Remark 6(a), $\mathbb{P}^R(t)$ is rational. Thus, by Theorem 39(b), $\mathbb{P}^Q(t)$ is rational. \square

5. APPLICATIONS: SHORT AND STRETCHED GORENSTEIN RINGS

Definition 42. *Let $(Q, \mathbf{m}_Q, \mathbf{k})$ be a Gorenstein Artin local ring with Hilbert function H_Q .*

- i) *We say that Q is a short Gorenstein ring if $H_Q = (1, h, n, 1)$, i.e., if $\mathbf{m}_Q^3 = \text{soc}(Q)$.*
- ii) *We say Q is a stretched Gorenstein ring if $H_Q = (1, h, 1, \dots, 1)$, i.e., \mathbf{m}_Q^2 is principal and $\mathbf{m}_Q^3 \neq 0$.*

In her paper on stretched Gorenstein rings, Sally proved a structure theorem ([14, Corollary 1.2]) for a stretched Gorenstein local ring $(Q, \mathfrak{m}_Q, \mathbf{k})$ when $\text{char}(\mathbf{k}) \neq 2$. Elias and Rossi proved a similar structure theorem ([7, Theorem 4.1]) for a short Gorenstein local ring $(Q, \mathfrak{m}_Q, \mathbf{k})$ when $\text{char}(\mathbf{k}) = 0$ and \mathbf{k} is algebraically closed. In particular, if Q is a \mathbf{k} -algebra where $\text{char}(\mathbf{k}) = 0$, then, if Q is either a short (with \mathbf{k} algebraically closed) or a stretched (with $\text{ll}(Q) \geq 3$) Gorenstein Artin \mathbf{k} -algebra, the structure theorems of Elias-Rossi and Sally respectively show that Q corresponds to a polynomial of the form $F(\underline{Y}) + Z_1^2 + \cdots + Z_n^2$ in terms of Macaulay's inverse systems. Thus, in the respective cases, this shows that Q is a connected sum, where one of the components is a Gorenstein ring with Loewy length equal to two.

Theorem 39 generalizes these two results, which can be seen as follows:

Proposition 43. *Let Q be either a short or a stretched Gorenstein ring. Then $G = \text{gr}(Q)$ is Gorenstein up to linear socle.*

Proof. Let Q_0 be the quotient of $G = \text{gr}(Q)$ as defined by Iarrobino (see Remark 3). Since the last two Hilbert coefficients of G and Q_0 are the same by Remark 3, and the Hilbert function of a graded Gorenstein \mathbf{k} -algebra is palindromic, we see the following:

- i) Let Q be a short Gorenstein ring with $H_Q = (1, h, n, 1)$. Then $H_{Q_0} = (1, n, n, 1)$.
- ii) Let Q be a stretched Gorenstein ring with $H_Q = (1, h, 1, \dots, 1)$. Then $H_{Q_0} = (1, 1, 1, \dots, 1)$.

Thus if we take A to be Q_0 in Proposition 34(ii), we see that G is Gorenstein up to linear socle. \square

Remark 44. *From the above proof and Remark 35, it is clear that if Q is a short Gorenstein ring with $H_Q = (1, h, n, 1)$, then $\text{edim}(Q) - \text{type}(G) = n - 1$ and if Q is a stretched Gorenstein ring with $H_Q = (1, h, 1, \dots, 1)$, then $\text{edim}(Q) = \text{type}(G)$.*

Theorem 45. *Let $(Q, \mathfrak{m}_Q, \mathbf{k})$ be a Gorenstein Artin \mathbf{k} -algebra with $\text{ll}(Q) \geq 3$. Then Q is stretched if and only if $G = \text{gr}(Q)$ is Gorenstein up to linear socle with $\text{edim}(Q) = \text{type}(G)$. In particular, Q is a connected sum with $\frac{1}{\mathbb{P}^Q(t)} = 1 - \text{edim}(Q)t + t^2$ when $\text{edim}(Q) \geq 2$.*

Proof. Assume $G = \text{gr}(Q)$ is Gorenstein up to linear socle with $\text{edim}(Q) = \text{type}(G)$. Then Proposition 34 and Remark 35 give that $G \simeq A \times_{\mathbf{k}} B$ with $\text{edim}(B) = \text{edim}(Q) - 1$ and $\text{edim}(A) = 1$. Since $\text{ll}(A) = \text{ll}(Q)$, we have Q is stretched by Theorem 39. The converse is a consequence of Proposition 43 and Remark 44.

Finally, the formula for $\frac{1}{\mathbb{P}^Q(t)}$ follows from Remark 44 and Remark 13. \square

Remark 46. *In [5], the authors show that if $(Q, \mathfrak{m}_Q, \mathbf{k})$ is a short Gorenstein Artin \mathbf{k} -algebra with Hilbert function $H_Q = (1, h, n, 1)$, then $\mathbb{P}^Q(t)$ is rational with the assumption that \mathbf{k} is an algebraically closed field of characteristic zero. In Theorem 47, we show the same is true for $n \leq 3$ without any restrictions on \mathbf{k} and for $n = 4$ with the extra assumption that $\text{char}(\mathbf{k}) \neq 2$.*

Theorem 47. *Let $(Q, \mathfrak{m}_Q, \mathbf{k})$ be a short Gorenstein Artin \mathbf{k} -algebra with Hilbert function $H_Q = (1, h, n, 1)$. Then Q is a connected sum. Furthermore, if $\text{char}(\mathbf{k}) \neq 2$ and $n = 4$, or $n \leq 3$, then $\mathbb{P}^Q(t)$ is rational.*

Proof. By Theorem 39 and Proposition 43, Q is a connected sum. The fact that $\mathbb{P}^Q(t)$ is rational follows from Remark 44 and Corollary 41. \square

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